Envelope of the family of curves involving two parameters :-

Let $\phi(x, y, c_1, c_2) = 0$ — (i) be the given family of curves involving c_1, c_2 are two parameters; related by $f(c_1, c_2) = 0$ — (ii)

Differentiating eqn (i) and (ii) w.r.t. c_1, partially, we have

$$\frac{\partial \phi}{\partial c_1} + \frac{\partial \phi}{\partial c_2} \cdot \frac{\partial c_2}{\partial c_1} = 0 \quad \text{— (iii)}$$

and $$\frac{\partial f}{\partial c_1} + \frac{\partial f}{\partial c_2} \cdot \frac{\partial c_2}{\partial c_1} = 0 \quad \text{— (iv)}$$

Now eliminating c_1, c_2 from eqns (i), (ii), (iii) & (iv) we obtained a relation only in x and y, and involving some fixed constant; which is required envelope of the eqn (i).

Example :- (1) Find the envelope of the curve $\frac{x}{a} + \frac{y}{b} = 1$; where a, b are related by $ab = c^2$.

Solution :- Given that family of curves $\frac{x}{a} + \frac{y}{b} = 1$ — (i)

and a, b are connected by $ab = c^2$ — (ii)

Here c is some fixed constant.

Partially differentiating (i) and (ii) w.r.t. a, we have

$$-\frac{x}{a^2} - \frac{y}{b^2} \cdot \frac{db}{da} = 0 \Rightarrow \frac{db}{da} = -\left(\frac{x}{y}\right) \cdot \left(\frac{b}{a}\right)^2 \quad \text{— (iii)}$$

and $$b + a \cdot \frac{db}{da} = 0 \Rightarrow \frac{db}{da} = -\frac{b}{a} \quad \text{— (iv)}$$

From (iii) & (iv); we have $\left(-\frac{b}{a}\right) = -\left(\frac{x}{y}\right) \cdot \left(\frac{b}{a}\right)^2$

$$\Rightarrow \frac{x}{a} = \frac{y}{b} \quad \text{— (v)}$$

eqns (v) using in eqn (i); we have $\frac{y}{b} + \frac{y}{b} = 1 \Rightarrow b = 2y$ (vi) using in (v) $b=2y$; we have $a=2x$. — (vii)

Using $a=2x$, $b=2y$, in eqn (ii)

We have

$$2x \cdot 2y = c^2$$

$$\Rightarrow \boxed{xy = \frac{c^2}{4}}$$

Which is required envelope.

Excercise:- (i) Find the envelope of $\frac{x}{a}+\frac{y}{b}=1$

when $a^m b^n = c^{m+n}$; where c is a constant.

Excercise (ii):- Find the envelope of $\left(\frac{x}{a}\right)^m + \left(\frac{y}{b}\right)^m = 1$

When $a^n + b^n = c^n$; where c is constant.

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